

# LOWER BOUND FOR DILATATIONS

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**ABSTRACT.** We prove a new lower bound for the dilatation of an arbitrary pseudo-Anosov map on a surface of genus  $g$  with  $n$  punctures. Our bound improves the former exponential dependence on the genus by a polynomial dependence.

## 1. INTRODUCTION

Let  $S = S_{g,n}$  be a surface of genus  $g$  with  $n$  punctures, where  $\chi(S) = 2 - 2g - n < 0$ . The mapping class group of  $S$ ,  $Mod(S_{g,n})$ , is the group of orientation preserving homeomorphisms of  $S$  up to isotopy. Here, the punctures are assumed to be fixed setwise by the homeomorphism. Thurston-Nielsen classification of mapping classes states that every mapping class is either pseudo-Anosov, reducible or finite order [10]. Pseudo-Anosovs are often the ones whose understanding is the crucial part in studying the mapping class group.

Associated to any pseudo-Anosov map is an algebraic integer called the dilatation or the stretch factor. The dilatation measures how much the map stretches/shrinks in the two canonical directions at each point of the surface. From dynamical point of view, the logarithm of the dilatation is the entropy of the pseudo-Anosov map. Ivanov proved that on a fixed surface, the set of dilatations is a discrete subset of  $(1, \infty)$  [5, 2]. In particular there exists a minimum dilatation. Let us denote by  $l_{g,n}$  the logarithm of the minimum dilatation for pseudo-Anosov maps on  $S_{g,n}$ . We use the notation  $l_{g,n}^+$  for the minimum dilatation for pseudo-Anosovs with orientable invariant foliations. Finding the minimum dilatation or its asymptotic has been of great importance. One motivation is that  $l_{g,n}$  is the systole (the length of the shortest geodesic) of the moduli space with the Teichmüller metric. Another motivation comes from the relation between low-dilatation pseudo-Anosov maps and low-volume fibered hyperbolic 3-manifolds [1]. Penner found the asymptotic of this number for closed surfaces [8]. He proved that there are constants  $c_1, c_2 > 0$  such that for any  $g \geq 2$

$$\frac{c_1}{g} \leq l_{g,0} \leq \frac{c_2}{g}.$$

Our aim is to understand the asymptotic of  $l_{g,n}$  similarly. Recall that Penner has proved the following [8]

$$l_{g,n} \geq \frac{\log(2)}{12g - 12 + 4n}.$$

which is comparable to  $\frac{1}{|\chi(S)|}$ , up to multiplicative constants. Tsai has obtained another lower bound for  $l_{g,n}$ , which gives a better bound than Penner's theorem when  $n$  is large compared to  $g$  [11]. Let  $\Gamma_S(3)$  denote the kernel of the action of  $Mod(S_{g,0})$  on  $H_1(S_{g,0}; \mathbb{Z}/3\mathbb{Z})$ . Define

$$\Theta(g) := [Mod(S) : \Gamma_S(3)].$$

Note that  $\Theta(g)$  is super-exponentially large in  $g$  [11]<sup>1</sup>.

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<sup>1</sup>In fact standard theorems imply that it is larger than  $3^{g^2}$ .

**Theorem 1.1.** (Tsai) For any  $g \geq 2$  and  $n \geq 0$  we have the following:

$$l_{g,n} \geq \min \left\{ \frac{1}{\Theta(g)} \frac{\log(2)}{(12g-12)}, \frac{1}{\Theta(g)} \frac{\log(3|\chi(S)|)}{6|\chi(S)|} \right\}.$$

Note that when  $n$  is large compared to  $g$ , the minimum is the second expression. The following theorem shows that one can replace  $\Theta(g)$  in Tsai's theorem by a term that is polynomially small in  $g$ .

**Theorem 1.2.** Given any positive real number  $\alpha$ , there exist a constant  $C > 0$  (depending on  $\alpha$ ) such that for any  $g \geq 2$  and  $n \geq 0$  we have the following:

$$l_{g,n} \geq \frac{C}{g^{2+\alpha}} \frac{\log(|\chi(S)|)}{|\chi(S)|}$$

and

$$l_{g,n}^+ \geq \frac{C}{g^{1+\alpha}} \frac{\log(|\chi(S)|)}{|\chi(S)|}.$$

**Conjecture 1.3.** There is a constant  $C > 0$  such that for all  $g \geq 2$  and  $n \geq 0$  we have the following:

$$l_{g,n} \geq \frac{C}{g} \frac{\log(|\chi(S)|)}{|\chi(S)|}.$$

Our lower bound should be compared with Tsai's upper bound for  $l_{g,n}$  [11]. Tsai proves that there is a constant  $C > 0$  such that for any  $g \geq 2$  and any  $n \geq 12g + 7$  the following holds:

$$l_{g,n} \leq Cg \frac{\log |\chi(S)|}{|\chi(S)|}.$$

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## 2. PROOF OF THE THEOREM

*Proof.* Let  $f \in \text{Mod}(S_{g,n})$  be a pseudo-Anosov map. Denote by  $\lambda(f)$  the dilatation of  $f$ . The idea is to look at the Lefschetz number of  $f$ , which we denote by  $L(f)$ . Define  $\hat{f} \in \text{Mod}(S_{g,0})$  to be the map obtained by forgetting the punctures. The following two observations have been made by Tsai [11].

1)  $L(f) = L(\hat{f})$ .

2) If  $L(f) < 0$  and  $f$  is pseudo-Anosov, then  $\log(\lambda(f)) \geq \frac{\log(3|\chi(S)|)}{6|\chi(S)|}$ .

The aim is to find a suitable power  $\nu$  of  $f$  such that  $L(f^\nu) < 0$  and then use the above bound. For any map  $\phi \in \text{Mod}(S_{g,0})$ , we have  $L(\phi) = 2 - \text{Tr}(\phi_*)$  where  $\phi_* : H_1(S) \rightarrow H_1(S)$  is the induced map on homology. Lemma 2.2 shows that one can find such a power that is at most polynomially large in terms of the genus.

In Lemma 2.2, take  $B = 2$ ,  $\phi = \hat{f}$  and  $A = \phi_*$ . Therefore  $m = 2g$ . Hence, if  $g \gg 0$  there is some  $\nu \leq (2g)^{2+\alpha}$  such that

$$L(f^\nu) = L(\widehat{f^\nu}) = L((\hat{f})^\nu) = 2 - \text{Tr}(A^\nu) < 0.$$

Since  $f^\nu$  is pseudo-Anosov we have the following

$$\begin{aligned} \nu \log(\lambda(f)) &= \log(\lambda(f^\nu)) \geq \frac{\log(3|\chi(S)|)}{6|\chi(S)|}. \\ \Rightarrow \log(\lambda(f)) &\geq \frac{1}{\nu} \frac{\log(3|\chi(S)|)}{6|\chi(S)|} \geq \frac{1}{(2g)^{2+\alpha}} \frac{\log(3|\chi(S)|)}{6|\chi(S)|}. \end{aligned}$$

This finishes the proof when  $g \gg 0$  let say for  $g \geq N$ . For the finitely many remaining values of  $2 \leq g < N$ , we use Lemma 2.3 [7]. Since  $\det(A) = 1$ , by Lemma 2.3 there exist a  $1 \leq \nu \leq 8^{2g}$  such that

$$\text{Tr}(A^\nu) \geq \frac{2g}{\sqrt{2}} > 2.$$

Therefore

$$\log(\lambda(f)) \geq \frac{1}{\nu} \frac{\log(3|\chi(S)|)}{6|\chi(S)|} \geq \frac{1}{8^{2g}} \frac{\log(3|\chi(S)|)}{6|\chi(S)|}.$$

So if we define  $c_j = \frac{j^{2+\alpha}}{8^{2j}}$  and set

$$C' = \min \left\{ c_1, \dots, c_{N-1}, \frac{1}{2^{2+\alpha}} \right\}.$$

Then, we have the following for each  $g \geq 2$  and  $n \geq 0$

$$\log(\lambda(f)) \geq \frac{C'}{g^{2+\alpha}} \frac{\log(3|\chi(S)|)}{6|\chi(S)|} \geq \frac{C}{g^{2+\alpha}} \frac{\log(|\chi(S)|)}{|\chi(S)|}$$

for  $C = \frac{C'}{6}$ .

Finally, if  $f$  has orientable invariant foliations, then the dilatation and the homological dilatation are equal (for a proof see [6]). Therefore, the spectral radius of  $A$  is larger than 1. Hence we can repeat the above argument with  $g^{1+\alpha}$  instead of  $g^{2+\alpha}$ .  $\square$

*Remark 2.1.* One can use Theorem 1.1 instead of lemma 2.3 to take care of the finitely many remaining values of  $g < N$ . However, we preferred to use a more elementary approach.

**Lemma 2.2.** *Fix  $B > 0$  and  $\epsilon > 0$ . For any  $m \gg 0$  (depending on  $B$  and  $\epsilon$ ) and any  $A \in SL(m, \mathbb{Z})$  we have the following: If the spectral radius of  $A$  is greater than 1 then there is some  $\nu$ ,  $1 \leq \nu \leq m^{1+\epsilon}$  such that*

$$\text{Tr}(A^\nu) > B.$$

*If the spectral radius of  $A$  is equal to 1 then there is some  $\nu$ ,  $1 \leq \nu \leq m^{2+\epsilon}$  such that*

$$\text{Tr}(A^\nu) > B.$$

**Lemma 2.3.** *Let  $z_1, \dots, z_m$  be complex numbers. Define*

$$S_\nu = z_1^\nu + \dots + z_m^\nu.$$

*There is a  $\nu$ ,  $1 \leq \nu \leq 8^m$  such that*

$$\text{Re}(S_\nu) \geq \frac{1}{\sqrt{2}} \sum_{j=1}^m |z_j|^\nu.$$

*In particular if  $|z_1 \dots z_m| = 1$  then there is a  $\nu$ ,  $1 \leq \nu \leq 8^m$  such that  $\text{Re}(S_\nu) \geq \frac{m}{\sqrt{2}}$ .*

**Proof of Lemma 2.2** We consider two cases:

**Case 1) If the spectral radius of  $A$  is larger than 1**

*Proof.* Recall the following theorems of Dobrowolsky [3] and Schinzel-Zassenhaus [9]

**Theorem 2.4.** (Dobrowolsky) *Let  $\lambda$  be an algebraic integer of degree  $d$  and define  $\overline{|\lambda|}$  to be the maximum modulus between all Galois conjugates of  $\lambda$ , including itself. For large enough  $d$  if  $\lambda$  is not a root of unity then*

$$\overline{|\lambda|} \geq 1 + \frac{2}{d} \left( \frac{\log \log(d)}{\log(d)} \right)^3.$$

**Theorem 2.5.** (Schinzel-Zassenhaus) *If an algebraic integer  $\lambda \neq 0$  is not a root of unity, and if  $2s$  among its conjugates are complex, then*

$$\overline{|\lambda|} > 1 + 4^{-s-2}.$$

The two theorems together imply that there exist a constant  $c > 0$  such that for all  $d$

$$\overline{|\lambda|} \geq 1 + \frac{c}{d \log(d)^3} \quad (*)$$

This is because by Dobrowolsky's theorem one can take  $c = 2$  for large  $d$ , say for  $d \geq M$ . For the finitely many remaining values of  $2 \leq d < M$  one can take  $c = 4^{-M-2}$ . Hence, in general  $c = \min \{2, 4^{-M-2}\}$  works.

Let  $\lambda_1, \dots, \lambda_m$  be the eigenvalues of  $A$  with  $\lambda_1$  having the maximum modulus between them. Therefore  $|\lambda_1| > 1$ . By the previous discussion we have the following:

$$|\lambda_1| \geq 1 + \frac{c}{d \log(d)^3} \geq 1 + \frac{c}{m \log(m)^3}.$$

Define  $z_j = \frac{\lambda_j}{|\lambda_1|}$ . Hence  $z_1, \dots, z_m$  are complex numbers with  $\max |z_j| = 1$ . Define

$$S_\nu = z_1^\nu + \dots + z_m^\nu.$$

In particular  $S_\nu$  is always a real number by Newton identities. Set  $K_0 = 20 \left( \frac{B}{c} \right) (m \log(m)^3)$ , where  $c$  is the constant in (\*). Set  $K = m^{1+\epsilon}$ . Note that for  $m \gg 0$  we have  $K \geq 5(m + 2BK_0)$ . We consider two cases

1) There exists  $1 \leq \nu \leq K_0$  such that  $S_\nu > B$ . Then

$$\text{Tr}(A^\nu) = |\lambda_1|^\nu S_\nu \geq S_\nu > B.$$

2) For each  $1 \leq \nu \leq K_0$  we have  $S_\nu \leq B$ . The proof in this case follows the lines of the proof of Cassel's theorem [7]. Let  $P(z) = \frac{1}{2} + \sum_{\nu=1}^K (1 - \frac{\nu}{K+1}) z^\nu$ . Then  $\text{Re}(P(z)) \geq 0$  whenever  $|z| \leq 1$  by the properties of the Fejer kernel. Let  $z_j = r_j e^{i\theta_j} := r_j e^{2\pi i \theta_j}$ . We have the following

$$\begin{aligned} \sum_{\nu=1}^K \left(1 - \frac{\nu}{K+1}\right) (1 + \cos 2\pi \nu \theta_1) \text{Re}(S_\nu) &= \sum_{j=1}^m \sum_{\nu=1}^K \left(1 - \frac{\nu}{K+1}\right) r_j^\nu (1 + \cos 2\pi \nu \theta_1) \cos 2\pi \nu \theta_j \\ &= \sum_{j=1}^m \text{Re} \left[ P(z_j) + \frac{1}{2} P(r_j e^{i(\theta_j - \theta_1)}) + \frac{1}{2} P(r_j e^{i(\theta_j + \theta_1)}) - 1 \right] \end{aligned}$$

Since  $P(r_1) = P(1) = \frac{K+1}{2}$ , we obtain that the above is

$$\geq \frac{K+1}{4} - m.$$

Now we have the following estimate

$$\begin{aligned} & \sum_{\nu=K_0}^K \left(1 - \frac{\nu}{K+1}\right) (1 + \cos 2\pi\nu\theta_1) \operatorname{Re}(S_\nu) = \\ & \sum_{\nu=1}^K \left(1 - \frac{\nu}{K+1}\right) (1 + \cos 2\pi\nu\theta_1) \operatorname{Re}(S_\nu) - \sum_{\nu=1}^{K_0-1} \left(1 - \frac{\nu}{K+1}\right) (1 + \cos 2\pi\nu\theta_1) \operatorname{Re}(S_\nu) \\ & \geq \frac{K+1}{4} - m - 2BK_0. \end{aligned}$$

On the other hand we have

$$\sum_{\nu=K_0}^K \left(1 - \frac{\nu}{K+1}\right) (1 + \cos 2\pi\nu\theta_1) \leq K.$$

Therefore, there exist  $K_0 \leq \nu \leq K$  such that

$$S_\nu \geq \frac{\frac{K+1}{4} - m - 2BK_0}{K} > \frac{1}{4} - \frac{m + 2BK_0}{K} \geq \frac{1}{4} - \frac{1}{5} = \frac{1}{20}.$$

Now we have

$$\begin{aligned} \operatorname{Tr}(A^\nu) &= |\lambda_1|^\nu S_\nu \geq \left(1 + \frac{c}{m \log(m)^3}\right)^{K_0} \times \frac{1}{20} \\ &\geq \left(1 + \frac{cK_0}{m \log(m)^3}\right) \times \frac{1}{20} > B. \end{aligned}$$

□

*Remark 2.6.* It follows from the proof that conditional on the Schinzel-Zassenhaus conjecture [9], one can replace the upper bound  $m^{1+\epsilon}$  for  $\nu$  by a linear bound (with linear constant just depending on  $B$ ).

**Conjecture 2.7.** (*Schinzel-Zassenhaus*) *There exists a constant  $c > 0$  such that for any algebraic integer  $\lambda$  of degree  $d$  which is not a root of unity we have*

$$|\overline{\lambda}| \geq 1 + \frac{c}{d}.$$

### Case 2) If the spectral radius of $A$ is equal to 1

*Proof.* Let  $Q(z)$  be the characteristic polynomial of  $A$ . By the assumption, all roots of  $Q$  have absolute value at most one. Recall the following theorem of Kronecker:

Let  $f$  be a monic polynomial with integer coefficients in  $z$ . If all roots of  $f$  have absolute value at most 1 then  $f$  is a product of cyclotomic polynomials and/or a power of  $z$ . Here, there can not be any power of  $z$ , since  $Q(0) = \det(A) = 1$ . So we can write  $Q$  as

$$Q(z) = \prod_{j=1}^l \Phi_{k_j}(z)$$

where  $\Phi_{k_j}(z)$  is the  $k_j$ -th cyclotomic polynomial and  $k_1 \leq k_2 \leq \dots \leq k_l$  are natural numbers. In particular, by comparing the degrees we deduce that

$$\varphi(k_1) + \dots + \varphi(k_l) = m$$

where  $\varphi$  is the Euler totient function. Take  $B'$  such that for each  $t > B'$  we have  $\varphi(t) > B$ . This is possible since  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . In fact more is true. For any  $\delta > 0$  we have [4]

$$\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t^{1-\delta}} = \infty.$$

Firstly, we specify how large  $m$  should be. We require that  $m > (B')!$ . Moreover assume that  $m$  is large enough so that for each  $t \geq m^{1+\epsilon}$  we have  $\varphi(t) > m + B$ . Note that  $k_l < m^{1+\epsilon}$  since  $\varphi(k_l) \leq m$ . We consider two cases

1)  $B' < k_l < m^{1+\epsilon}$ . Let  $g(z)$  be the polynomial whose roots are the  $k_l$ -powers of the roots of  $\frac{Q(z)}{\Phi_{k_l}(z)}$  allowing repetitions. Hence,  $g$  has integer coefficients by Newton's identities and  $\deg(g) < m$ . Let  $\nu = k_l \cdot \nu'$  where  $1 \leq \nu' \leq m$  is chosen such that the sum,  $S$ , of the  $\nu'$ -powers of the roots of  $g$  is non-negative. Such a  $\nu'$  exists by the following lemma from [7].

**Lemma 2.8.** *Let  $z_1, \dots, z_n$  be all the roots of a polynomial with real coefficients. Define*

$$S_\nu = z_1^\nu + \dots + z_n^\nu$$

*Then  $S_\nu \geq 0$  for some integer  $\nu$  in the range  $1 \leq \nu \leq n + 1$ .*

Since  $\varphi(k_l) > B$  we have the following

$$\text{Tr}(A^\nu) = \varphi(k_l) + S > B.$$

Note that  $\nu = k_l \cdot \nu'$  is at most  $m^{1+\epsilon} \cdot m = m^{2+\epsilon}$ .

2)  $k_l \leq B'$ . Take  $\nu = (B')!$ . Then

$$\text{Tr}(A^\nu) = m > B.$$

This completes the proof.  $\square$

**Proof of Lemma 2.8** We closely follow the proof from [7]. Let  $\sigma_j$  be the  $j$ -th elementary symmetric function of  $z_1, \dots, z_n$ . Therefore,  $\sigma_j$  is real for each  $1 \leq j \leq n$ . Recall the Newton-Girard identities

$$r\sigma_r = \sum_{\nu=1}^r (-1)^{\nu-1} \sigma_{r-\nu} S_\nu$$

for  $1 \leq r \leq n$ . Suppose that  $S_\nu < 0$  for  $1 \leq \nu \leq n$ . Using Newton-Girard identities and induction we deduce that  $(-1)^j \sigma_j > 0$  for  $1 \leq j \leq n$ . On the other hand, another set of Newton-Girard identities state that

$$S_{t+n+1} = \sum_{\nu=t+1}^{t+n} S_\nu (-1)^{t+n-\nu} \sigma_{t+n+1-\nu}$$

for  $t \geq 0$ . Putting  $t = 0$  we see that  $S_\nu < 0$  and  $(-1)^{n-\nu} \sigma_{n+1-\nu} < 0$  for  $1 \leq \nu \leq n$ , therefore all summands on the right hand side are positive. Hence  $S_{n+1} > 0$ .

**Proof of Lemma 2.3** Decompose the plane into 8 equal sections according to the angle. For  $1 \leq i \leq 8$ :

$$V_i = \{(r, \theta) \in \mathbb{R}^2 \mid (i-1)\frac{2\pi}{8} \leq \theta < i\frac{2\pi}{8}\}.$$

For each  $1 \leq k \leq 8^m + 1$  we code the regions in which the points  $z_1^k, \dots, z_m^k$  lie with a vector

$$A_k = (a_1, \dots, a_m)$$

where  $1 \leq a_i \leq 8$ . By the pigeonhole principle there are distinct indices  $1 \leq i, j \leq 8^m + 1$  such that  $A_i = A_j$ . Therefore

$$A_{|j-i|} = (b_1, \dots, b_m)$$

where  $b_\ell \in \{1, 8\}$  for each  $1 \leq \ell \leq m$ . This implies that for  $\nu = |j - i|$

$$Re(S_\nu) \geq \frac{1}{\sqrt{2}} \sum_{j=1}^m |z_j|^\nu.$$

The conclusion of the second part of the lemma is obtained by using the AM-GM inequality:

$$\frac{|z_1|^\nu + \dots + |z_m|^\nu}{m} \geq \sqrt[m]{|z_1 \dots z_m|^\nu} = 1.$$

**Question 2.9.** Find an explicit function  $f(g, n)$  such that there exist constants  $c_1, c_2 > 0$  with

$$c_1 f(g, n) \leq l_{g,n} \leq c_2 f(g, n)$$

for all  $g \geq 2$  and  $n \geq 0$ .

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